

CHARACTER INNER AMENABILITY OF CERTAIN BANACH ALGEBRAS

H.R. EBRAHIMI VISHKI AND A.R. KHODDAMI

ABSTRACT. Character inner amenability for certain class of Banach algebras consist of projective tensor product $A \hat{\otimes} B$, Lau product $A \times_{\theta} B$ and module extension $A \oplus X$ are investigated. Some illuminating examples are also included.

1. INTRODUCTION

The concept of left amenability for a Lau algebra (a predual of a von Neumann algebra for which the identity of the dual is a multiplicative linear functional, [6]) has been extensively extended for an arbitrary Banach algebra by introducing the notion of φ -amenability in Kaniuth *et al.* [4]. A Banach algebra A was called φ -amenable ($\varphi \in \Delta(A)$ = the spectrum of A) if there exists a $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ ($a \in A$, $f \in A^*$). A was called character amenable if it is φ -amenable for each $\varphi \in \Delta(A)$. Many aspects of φ -amenability have been investigated in [5, 9, 2]. Recently Jabbari *et al.* [3] have introduced the φ -version of inner amenability. A Banach algebra A was said to be φ -inner amenable if there exists a $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = m(a \cdot f)$ ($f \in A^*$, $a \in A$). Such a m will sometimes be referred to as a φ -inner mean and A is said to be character inner amenable if and only if A is φ -inner amenable for every $\varphi \in \Delta(A)$. As they have remarked in [3, Remark 2.4], this concept considerably generalizes the notion of inner amenability for Lau algebras which introduced by Nasr-Isfahani [7]. They also gave several characterizations of φ -inner amenability. For instance, as in the case of φ -amenability in [4, Theorem 1.4], they have shown that a φ -inner mean is in fact some w^* -cluster point of a bounded net (a_{α}) in A satisfying $\|a_{\alpha}a - aa_{\alpha}\| \rightarrow 0$, for all $a \in A$ and $\varphi(a_{\alpha}) = 1$ for all α ; [3, Theorem 2.1].

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In this paper, we are going to investigate the character inner amenability for certain products of Banach algebras consist of projective tensor product $A \hat{\otimes} B$, Lau product $A \times_{\theta} B$ and the module extension $A \oplus X$. For instance, we show that the projective tensor product $A \hat{\otimes} B$ is character inner amenable if and only if both A and B enjoy the same property.

2. PRELIMINARY RESULTS AND EXAMPLES

Before we proceed for the results we need a bit preliminaries. The second dual A^{**} of a Banach algebra A can be made into Banach algebra under each of Arens products \square and \diamond which are defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$,

$$\langle m \square n, f \rangle = \langle m, n \cdot f \rangle, \langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle, \langle f \cdot a, b \rangle = \langle f, ab \rangle;$$

and

$$\langle f, m \diamond n \rangle = \langle f \cdot m, n \rangle, \langle a, f \cdot m \rangle = \langle a \cdot f, m \rangle, \langle b, a \cdot f \rangle = \langle ba, f \rangle.$$

We commence with the next definition from [3].

Definition 2.1. Let A be a Banach algebra and let $\varphi \in \Delta(A)$. Then A is called φ -inner amenable if there exists a $m \in A^{**}$ such that $m(\varphi) = 1$ and $m \square a = a \square m$, ($a \in A$). We call such a m a φ -inner mean. A Banach algebra A is called character inner amenable if A is φ -inner amenable for all $\varphi \in \Delta(A)$.

The next straightforward characterization of φ -inner amenability (see [3, Theorem 2.1] which is inspired from [4, Theorem 1.4]) will be frequently used in the sequel.

Proposition 2.2. Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following statements are equivalent.

- (i) A is φ -inner amenable.
- (ii) There exists a bounded net (a_{α}) in A such that $\|aa_{\alpha} - a_{\alpha}a\| \rightarrow 0$ for all $a \in A$ and $\varphi(a_{\alpha}) = 1$ for all α .
- (iii) There exists a bounded net (a_{α}) in A such that $\|aa_{\alpha} - a_{\alpha}a\| \rightarrow 0$ for all $a \in A$ and $\varphi(a_{\alpha}) \rightarrow 1$.

Examples 2.3. (i) Every Banach algebra with a bounded approximate identity (e_{α}) is character inner amenable. Indeed, one can verify that $\|ae_{\alpha} - e_{\alpha}a\| \rightarrow 0$ and $\varphi(e_{\alpha}) \rightarrow 1$, for each $\varphi \in \Delta(A)$.

(ii) Every commutative Banach algebra is character inner amenable.

(iii) Let $A = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C} \right\}$ and define $\varphi : A \rightarrow \mathbb{C}$ by $\varphi\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = b$. A direct verification reveals that there is no bounded net (a_α) in A satisfying Proposition 2.2. Therefore A is not φ -inner amenable.

(iv) Given a Banach space A and fix a non-zero $\varphi \in A^*$ with $\|\varphi\| \leq 1$. Then the product $a \cdot b = \varphi(a)b$ turning A into a Banach algebra with $\Delta(A) = \{\varphi\}$. Trivially A has a left identity (indeed, every $e \in A$ with $\varphi(e) = 1$ is a left identity), while it has no bounded approximate identity in the case where $\dim(A) > 1$. In this case A is not φ -inner amenable. Indeed, if m is a φ -inner mean for A then $m(\varphi) = 1$ and $m \square a = a \square m$ for all $a \in A$. But a simple calculation reveals that $m \square a = m(\varphi)a$ and $a \square m = \varphi(a)m$. Therefore $a = \varphi(a)m$ for each $a \in A$, that is, $\dim(A) = 1$.

(v) Let A be the Banach algebra posed in (iv) which is generated by two elements a and b , that is, $\dim(A) = 2$ and let $\varphi \in A^*$ be so that $\varphi(a) = 1$ and $\varphi(b) = 0$. If I is the subspace generated by b then I is a closed ideal for which I and A/I are character inner amenable; however, A itself is not.

Note that, as [3, Theorem 2.8] demonstrates, if A is character inner amenable then so is A/I for each closed ideal I of A . However, I may not be character inner amenable; for example, the unitization of a non-character inner amenable Banach algebra is character inner amenable.

(vi) For a reflexive Banach algebra A with $\varphi \in \Delta(A)$ it is easy to verify that A is φ -inner amenable if and only if $Z(A) \cap (A - \ker \varphi) \neq \emptyset$, where $Z(A)$ is the algebraic center of A .

3. PROJECTIVE TENSOR PRODUCT $A \hat{\otimes} B$

Let $A \hat{\otimes} B$ be the projective tensor product of two Banach algebras A and B . For $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A \hat{\otimes} B)^*$ satisfying, $(f \otimes g)(a \otimes b) = f(a)g(b)$ ($a \in A, b \in B$). Recall that,

$$\Delta(A \hat{\otimes} B) = \{\varphi \otimes \psi, \varphi \in \Delta(A), \psi \in \Delta(B)\}.$$

In the next result, as in the case of character amenability in [4, Theorem 3.3], we investigate the character inner amenability of $A \hat{\otimes} B$. It is worthwhile mentioning that our

method of proof provides an alternative proof for [4, Theorem 3.3] which does not rely on derivation techniques.

Theorem 3.1. *Let A and B be Banach algebras and let $\varphi \in \Delta(A)$, $\psi \in \Delta(B)$. Then $A \hat{\otimes} B$ is $(\varphi \otimes \psi)$ -inner amenable if and only if A is φ -inner amenable and B is ψ -inner amenable. In particular, $A \hat{\otimes} B$ is character inner amenable if and only if both A and B are character inner amenable.*

Proof. Let $m \in (A \hat{\otimes} B)^{**}$ be a $\varphi \otimes \psi$ -inner mean. So $m(\varphi \otimes \psi) = 1$ and

$$m((f \otimes \psi) \cdot (a \otimes b)) = m((a \otimes b) \cdot (f \otimes \psi)), \quad (f \in A^*, a \in A, b \in B).$$

Define $m_\varphi : A^* \rightarrow \mathbb{C}$ such that $m_\varphi(f) = m(f \otimes \psi)$. So $m_\varphi(\varphi) = m(\varphi \otimes \psi) = 1$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$ and let $f \in A^*$ and $a \in A$. So

$$\begin{aligned} m_\varphi(f \cdot a) &= m(f \cdot a \otimes \psi) = m(f \cdot a \otimes \psi \cdot b_0) \\ &= m((f \otimes \psi) \cdot (a \otimes b_0)) = m((a \otimes b_0) \cdot (f \otimes \psi)) \\ &= m(a \cdot f \otimes b_0 \cdot \psi) = m(a \cdot f \otimes \psi) \\ &= m_\varphi(a \cdot f). \end{aligned}$$

It follows that A is φ -inner amenable, and similarly B is ψ -inner amenable.

For the converse let A be φ -inner amenable and B ψ -inner amenable. Then there exist bounded nets (a_α) in A and (b_β) in B such that $\varphi(a_\alpha) = 1$, $\|aa_\alpha - a_\alpha a\| \rightarrow 0$, ($a \in A$) and $\psi(b_\beta) = 1$, $\|bb_\beta - b_\beta b\| \rightarrow 0$, ($b \in B$). Consider the bounded net $(a_\alpha \otimes b_\beta)$ in $A \hat{\otimes} B$. So $(\varphi \otimes \psi)(a_\alpha \otimes b_\beta) = \varphi(a_\alpha)\psi(b_\beta) = 1$. Now let $\|a_\alpha\| \leq M_1$, $\|b_\beta\| \leq M_2$ and let $F = \sum_{j=1}^N c_j \otimes d_j \in A \otimes B$.

$$\begin{aligned} \|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| &= \left\| \sum_{j=1}^N [(c_j a_\alpha - a_\alpha c_j) \otimes d_j b_\beta + a_\alpha c_j \otimes (d_j b_\beta - b_\beta d_j)] \right\| \\ &\leq \sum_{j=1}^N M_2 \|d_j\| \|c_j a_\alpha - a_\alpha c_j\| + \sum_{j=1}^N M_1 \|c_j\| \|d_j b_\beta - b_\beta d_j\|. \end{aligned}$$

Since $\|c_j a_\alpha - a_\alpha c_j\| \rightarrow 0$ and $\|d_j b_\beta - b_\beta d_j\| \rightarrow 0$, ($1 \leq j \leq N$), so $\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| \rightarrow 0$.

Now let $w \in A \hat{\otimes} B$, so there exist sequences $\{c_j\} \subseteq A$ and $\{d_j\} \subseteq B$ such that $w = \sum_{j=1}^{\infty} c_j \otimes d_j$ with $\sum_{j=1}^{\infty} \|c_j\| \|d_j\| < \infty$. Let $\epsilon > 0$ be given, we choose $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} \|c_j\| \|d_j\| < \epsilon/4M_1M_2$. Put $F = \sum_{j=1}^N c_j \otimes d_j$. As $\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| \rightarrow 0$, so there exists (α_0, β_0) such that $\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| < \epsilon/2$ for all $(\alpha, \beta) \geq (\alpha_0, \beta_0)$. Now for such a (α, β) ,

$$\begin{aligned} \|w(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)w\| &= \|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F + \sum_{j=N+1}^{\infty} [c_j a_\alpha \otimes d_j b_\beta - a_\alpha c_j \otimes b_\beta d_j]\| \\ &\leq \|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| + 2M_1M_2 \sum_{j=N+1}^{\infty} \|c_j\| \|d_j\| \\ &< \epsilon/2 + 2M_1M_2 \cdot \epsilon/4M_1M_2 \\ &= \epsilon. \end{aligned}$$

Hence $\|w(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)w\| \rightarrow 0$. Applying Proposition 2.2 shows that $A \hat{\otimes} B$ is $(\varphi \otimes \psi)$ -inner amenable. \square

4. THE LAU PRODUCT $A \times_\theta B$

Let A and B be two Banach algebras with $\Delta(B) \neq \emptyset$. For a $\theta \in \Delta(B)$ the θ -Lau product $A \times_\theta B$ is defined as the cartesian product $A \times B$ with the algebra multiplication $(a, b) \cdot (c, d) = (ac + \theta(d)a + \theta(b)c, bd)$ and with the norm $\|(a, b)\| = \|a\| + \|b\|$.

This product was first introduced by Lau [6] for Lau algebras and followed by Sangani Monfared [8] for the general case. $A \times_\theta B$ is a Banach algebra and it is shown in [8, Proposition 2.4] that

$$\Delta(A \times_\theta B) = (\Delta(A) \times \{\theta\}) \cup (\{0\} \times \Delta(B)).$$

In a natural way the dual space $(A \times_\theta B)^*$ can be identified with $A^* \times B^*$ via $(f, g)((a, b)) = f(a) + g(b)$. Recall that the dual norm on $A^* \times B^*$ is $\|(f, g)\| = \max\{\|f\|, \|g\|\}$. Also if A^{**}, B^{**} and $(A \times_\theta B)^{**}$ are equipped with their first Arens products then $(A \times_\theta B)^{**} = A^{**} \times_\theta B^{**}$ as an isometric isomorphism. Also for $(m, n), (p, q) \in (A \times_\theta B)^{**}$ we have $(m, n) \square (p, q) = (m \square p + n(\theta)p + q(\theta)m, n \square q)$; see [8, Proposition 2.12].

The next result, which extends [7, Proposition 4.2], studies character inner amenability of $A \times_\theta B$.

Theorem 4.1. *Let $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$. Then*

- (i) *$A \times_\theta B$ is (φ, θ) -inner amenable if and only if either A is φ -inner amenable or B is θ -inner amenable.*
- (ii) *$A \times_\theta B$ is $(0, \psi)$ -inner amenable if and only if B is ψ -inner amenable.*
- (iii) *$A \times_\theta B$ is character inner amenable if and only if B is character inner amenable.*

Proof. (i) Let $A \times_\theta B$ be (φ, θ) -inner amenable. Then there exists $(m, n) \in \mathcal{A}^{**} \times_\theta B^{**}$ such that $(m, n)((\varphi, \theta)) = 1$ and $(m, n) \square (a, b) = (a, b) \square (m, n)$, for all $(a, b) \in A \times_\theta B$. It follows that $m(\varphi) + n(\theta) = 1$, $m \square a = a \square m$ and $n \square b = b \square n$ for all $a \in A$ and $b \in B$. Now if $n(\theta) = 0$ then $m(\varphi) = 1$ and so m is a φ -inner mean for A . If $n(\theta) \neq 0$ then $\frac{n}{n(\theta)} \square b = b \square \frac{n}{n(\theta)}$, that is, $\frac{n}{n(\theta)}$ is a θ -inner mean for B .

For the converse, suppose that m is a φ -inner mean for A then trivially $(m, 0)$ is a (φ, θ) -inner mean for $A \times_\theta B$. The same argument needs for the case that B is a ψ -inner amenable. (ii) needs a similar proof and (iii) follows trivially from (i) and (ii). \square

Now we turn our attention to the question of character inner amenability of the Banach algebras $A \oplus_\infty B$ and $A \oplus_p B$. Recall that these are equipped with the usual direct product multiplications and the norms $\|(a, b)\| = \max \{\|a\|, \|b\|\}$ and $\|(a, b)\| = (\|a\|^p + \|b\|^p)^{\frac{1}{p}}$, respectively. A direct verification shows that

$$\Delta(A \oplus_p B) = (\Delta(A) \times \{0\}) \cup (\{0\} \times \Delta(B)), \quad 1 \leq p \leq \infty;$$

from which we get the next result.

Proposition 4.2. *Let A and B be Banach algebras and let $1 \leq p \leq \infty$. Then $A \oplus_p B$ is character inner amenable if and only if both A and B are character inner amenable.*

5. MODULE EXTENSION AND TRIANGULAR BANACH ALGEBRAS

For a Banach algebra A and a Banach A -module X let $A \oplus X$ be the module extension Banach algebra which is equipped with the algebra product $(a, x) \cdot (b, y) = (ab, ay + xb)$, $(a, b \in A, x, y \in X)$ and the norm $\|(a, x)\| = \|a\| + \|x\|$. The second dual $(A \oplus X)^{**}$ can be identified with $A^{**} \oplus_1 X^{**}$ as a Banach space, and it is not difficult to verify that the first Arens product on $(A \oplus X)^{**}$ is given by $(m, \lambda) \square (n, \mu) = (m \square n, m\mu + \lambda n)$. Some aspects of the module extension Banach algebras have been discussed in [10].

Let A and B be Banach algebras and let X be a Banach A, B -module; that is, a left A -module and a right B -module satisfying $\|axb\| \leq \|a\|\|x\|\|b\|$, ($a \in A, b \in B, x \in X$). The corresponding triangular Banach algebra

$$\tau = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a \in A, x \in X, b \in B \right\}.$$

is equipped with the usual 2×2 -matrix operations and the norm $\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|x\| + \|b\|$. The Arens products on the second dual of τ is studied in [1]. Recall that the class of module extension Banach algebras includes the triangular Banach algebras. Indeed, τ can be identified with the module extension $(A \oplus_1 B) \oplus X$; in which X is considered as a $A \oplus_1 B$ -module under the operations $(a, b) \cdot x = ax$ and $x \cdot (a, b) = xb$.

Proposition 5.1. *Let A be a Banach algebra and X be a Banach A -module. Then for the module extension Banach algebra $A \oplus X$, $\Delta(A \oplus X) = \Delta(A) \times \{0\}$. In particular, for the triangular Banach algebra τ , $\Delta(\tau) = \Delta(A \oplus_1 B) \times \{0\}$.*

Proof. Trivially $\Delta(A) \times \{0\} \subseteq \Delta(A \oplus X)$. Let $(\varphi, \psi) \in \Delta(A \oplus X)$. So for $a, b \in A$, $(\varphi, \psi)((a, 0)(b, 0)) = (\varphi, \psi)((a, 0))(\varphi, \psi)((b, 0))$. It follows that $\varphi(ab) = \varphi(a)\varphi(b)$. Also for $x, y \in X$, $0 = (\varphi, \psi)((0, x)(0, y)) = (\varphi, \psi)((0, x))(\varphi, \psi)((0, y)) = \psi(x)\psi(y)$. So $\psi = 0$ and finally $\varphi \in \Delta(A)$. Hence $\Delta(A \oplus X) = \Delta(A) \times \{0\}$. The second part is clear. \square

The next result which studies the character amenability of $A \oplus X$ and τ is a direct application of [4, Theorem 1.4] for the module extension $A \oplus X$.

Proposition 5.2. *Let A be a Banach algebra, X be a Banach A -module and let $\varphi \in \Delta(A)$. Then $A \oplus X$ is $(\varphi, 0)$ -amenable if and only if there exists a bounded net (a_α, x_α) in $A \oplus X$ satisfying*

$$(i) \|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0 \text{ for all } a \in A \text{ and } \varphi(a_\alpha) = 1 \text{ for all } \alpha,$$

$$(ii) \|ax_\alpha - \varphi(a)x_\alpha\| \rightarrow 0 \text{ for all } a \in A \text{ and}$$

$$(iii) \|xa_\alpha\| \rightarrow 0, \text{ for all } x \in X.$$

Corollary 5.3. *(i) If $A \oplus X$ is character amenable then so is A . The converse also holds in the case where $XA = 0$.*

(ii) If τ is character amenable then both A and B are character amenable. The converse also holds in the case where $XB = 0$.

Similar to Proposition 5.2 we have the next result, which is based on Proposition 2.2, characterizing the character inner amenability of $A \oplus X$.

Proposition 5.4. *Let A be a Banach algebra, X be a Banach A -module and let $\varphi \in \Delta(A)$. Then $A \oplus X$ is $(\varphi, 0)$ -inner amenable if and only if there exists a bounded net (a_α, x_α) in $A \oplus X$ satisfying*

- (i) $\|aa_\alpha - a_\alpha a\| \rightarrow 0$ for all $a \in A$ and $\varphi(a_\alpha) = 1$ for all α ,
- (ii) $\|xa_\alpha - a_\alpha x\| \rightarrow 0$ for all $x \in X$, and
- (iii) $\|ax_\alpha - x_\alpha a\| \rightarrow 0$ for all $a \in A$.

Corollary 5.5. *If $A \oplus X$ is character inner amenable then A is character inner amenable. In particular if τ is character inner amenable then both A and B are character inner amenable.*

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DEPARTMENT OF PURE MATHEMATICS AND CENTRE OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD 91775, IRAN

E-mail address: `Vishki@um.ac.ir`

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD 91775, IRAN

E-mail address: `khoddami.alireza@yahoo.com`